

**RIESZ TRANSFORMS CHARACTERIZATIONS
OF HARDY SPACES H^1
FOR THE RATIONAL DUNKL SETTING
AND MULTIDIMENSIONAL BESSEL OPERATORS**

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ABSTRACT. We characterize the Hardy space H^1 in the rational Dunkl setting associated with the reflection group \mathbb{Z}_2^n by means of Riesz transforms. As a corollary we obtain a Riesz transform characterization of H^1 for product of Bessel operators in $(0, \infty)^n$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The theory of Dunkl operators had its origin in a series of seminal works [5]–[8] and was developed by many mathematicians afterwards. The Dunkl operators form a commuting system of differential-difference operators associated with a finite group of reflections. We refer the reader to the lecture notes [18, 19] and references therein for the rational Dunkl theory and to [16] for the trigonometric Dunkl theory.

In the present paper, on the Euclidean space \mathbb{R}^n , $n \geq 1$, we consider the Dunkl operators

$$D_j f(\mathbf{x}) = \frac{\partial}{\partial x_j} f(\mathbf{x}) + \frac{k_j}{x_j} [f(\mathbf{x}) - f(\sigma_j \mathbf{x})] \quad (j=1, 2, \dots, n)$$

associated with the reflections

$$(1.1) \quad \sigma_j(x_1, x_2, \dots, x_j, \dots, x_n) = (x_1, x_2, \dots, -x_j, \dots, x_n)$$

and the multiplicities $k_j \geq 0$. Their joint eigenfunctions form the Dunkl kernel

$$(1.2) \quad \mathbf{E}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^n E_{k_j}(x_j, y_j),$$

$$(1.3) \quad D_j \mathbf{E}(\cdot, \mathbf{y})(\mathbf{x}) = y_j \mathbf{E}(\mathbf{x}, \mathbf{y}),$$

where

$$(1.4) \quad \begin{aligned} E_k(x, y) &= \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})} \int_{-1}^{+1} (1-u)^{k-1} (1+u)^k e^{xyu} du \\ &= 2^{k-1/2} \Gamma\left(k + \frac{1}{2}\right) |xy|^{\frac{1}{2}-k} \left(I_{k-1/2}(|xy|) + \operatorname{sgn}(xy) I_{k+1/2}(|xy|) \right) \end{aligned}$$

(see for instance [18, p. 107, Example 2.1]). Here $I_\nu(x)$ is the modified Bessel function (see, e.g., [15, 24]). Notice that $\mathbf{E}(\mathbf{x}, \mathbf{y}) = e^{\langle \mathbf{x}, \mathbf{y} \rangle}$ if all multiplicities k_j vanish.

The Dunkl Laplacian

$$\mathbf{L}f(\mathbf{x}) = \sum_{j=1}^n D_j^2 f(\mathbf{x}) = \sum_{j=1}^n \left\{ \left(\frac{\partial}{\partial x_j} \right)^2 f(\mathbf{x}) + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} f(\mathbf{x}) - \frac{k_j}{x_j^2} [f(\mathbf{x}) - f(\sigma_j \mathbf{x})] \right\}$$

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is the infinitesimal generator of the heat semigroup $\{e^{t\mathbf{L}}\}_{t>0}$, which acts by linear self-adjoint operators on $L^2(\mathbb{R}^n, d\boldsymbol{\mu})$ and by linear contractions on $L^p(\mathbb{R}^n, d\boldsymbol{\mu})$, for every $1 \leq p \leq \infty$, where

$$(1.5) \quad d\boldsymbol{\mu}(\mathbf{x}) = d\mu_1(x_1) \dots d\mu_n(x_n) = |x_1|^{2k_1} \dots |x_n|^{2k_n} dx_1 \dots dx_n.$$

Clearly,

$$\mathbf{L}\mathbf{E}(\cdot, \mathbf{y})(\mathbf{x}) = |\mathbf{y}|^2 \mathbf{E}(\mathbf{x}, \mathbf{y}).$$

The heat semigroup, which is strongly continuous on $L^p(\mathbb{R}^n, d\boldsymbol{\mu})$ for $1 \leq p < \infty$, consists of integral operators

$$e^{t\mathbf{L}}f(\mathbf{x}) = \int_{\mathbb{R}^n} \mathbf{h}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\boldsymbol{\mu}(\mathbf{y})$$

associated with the heat kernel

$$(1.6) \quad \mathbf{h}_t(\mathbf{x}, \mathbf{y}) = \mathbf{c}_{\mathbf{k}}^{-1} t^{-\frac{\mathbf{N}}{2}} e^{-\frac{|\mathbf{x}|^2 + |\mathbf{y}|^2}{4t}} \mathbf{E}\left(\frac{\mathbf{x}}{\sqrt{2t}}, \frac{\mathbf{y}}{\sqrt{2t}}\right),$$

see, e.g., [17], where

$$(1.7) \quad \mathbf{N} = n + \sum_{j=1}^n 2k_j$$

is the homogeneous dimension and

$$\mathbf{c}_{\mathbf{k}} = 2^{\frac{\mathbf{N}}{2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}|^2}{2}} d\boldsymbol{\mu}(\mathbf{x}) = 2^{\mathbf{N}} \prod_{j=1}^n \Gamma(k_j + \frac{1}{2}).$$

The Dunkl transform is defined by

$$(1.8) \quad \mathcal{F}f(\boldsymbol{\xi}) = \mathbf{c}_{\mathbf{k}}^{-1} \int_{\mathbb{R}^n} f(\mathbf{x}) \mathbf{E}(\mathbf{x}, -i\boldsymbol{\xi}) d\boldsymbol{\mu}(\mathbf{x}).$$

It is an isometric isomorphism of $L^2(\mathbb{R}^n, d\boldsymbol{\mu})$ onto itself with the inversion formula:

$$f(\mathbf{x}) = \mathcal{F}^2 f(-\mathbf{x})$$

(see, e.g., [8], [12]).

The Hardy space $H_{\max, \mathbf{L}}^1$ associated with \mathbf{L} is the set of all functions $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ whose maximal heat function

$$(1.9) \quad \mathbf{h}_*f(\mathbf{x}) = \sup_{t>0} \left| \int_{\mathbb{R}^n} \mathbf{h}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\boldsymbol{\mu}(\mathbf{y}) \right|$$

belongs to $L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ and the norm is given by

$$\|f\|_{H_{\max, \mathbf{L}}^1} = \|\mathbf{h}_*f\|_{L^1(\mathbb{R}^n, d\boldsymbol{\mu})}.$$

Now we turn to the atomic definition of the Hardy space H^1 . Notice that \mathbb{R}^n , equipped with the Euclidean distance $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ and with the measure $\boldsymbol{\mu}$, is a space of homogeneous type in the sense of Coifman-Weiss [4]. An atom is a measurable function $a : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

- a is supported in a ball B ,
- $\|a\|_{L^\infty} \lesssim \boldsymbol{\mu}(B)^{-1}$,
- $\int_{\mathbb{R}^n} a(\mathbf{x}) d\boldsymbol{\mu}(\mathbf{x}) = 0$.

By definition, the atomic Hardy space H_{atom}^1 consists of all functions $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ which can be written as $f = \sum_{\ell} \lambda_{\ell} a_{\ell}$, where the a_{ℓ} 's are atoms and $\sum_{\ell} |\lambda_{\ell}| < +\infty$, and the norm is given by

$$\|f\|_{H_{\text{atom}}^1} = \inf \sum_{\ell} |\lambda_{\ell}|,$$

where the infimum is taken over all atomic decompositions of f .

Hardy spaces on spaces of homogeneous type (see, e.g., [4], [13], [23]) are extensions of the classical real Hardy spaces on \mathbb{R}^n . For characterizations and properties of the classical Hardy spaces we refer the reader to the original works [2], [11], [22], [3]. More information are given in the book [20] and references therein.

Hardy spaces associated with the Dunkl operator \mathbf{L} were studied in [1]. The following theorem was proved there.

Theorem 1.10. *The spaces $H_{\max, \mathbf{L}}^1$ and H_{atom}^1 coincide and the norms $\|f\|_{H_{\max, \mathbf{L}}^1}$ and $\|f\|_{H_{\text{atom}}^1}$ are equivalent.*

The present paper is a continuation of [1] and deals with the Riesz transforms characterization of $H_{\max, \mathbf{L}}^1$. We define the Riesz transforms in the Dunkl setting putting

$$\mathcal{R}_j = D_j(-\mathbf{L})^{-1/2}.$$

The operators \mathcal{R}_j can be expressed as the Dunkl multiplier operators, namely,

$$\begin{aligned} \mathcal{R}_j f(\mathbf{x}) &= D_j(-\mathbf{L})^{-1/2} f(\mathbf{x}) = D_j \int_{\mathbb{R}^n} \frac{1}{|\boldsymbol{\xi}|} \mathbf{E}(\mathbf{x}, i\boldsymbol{\xi}) \mathcal{F}f(\boldsymbol{\xi}) d\boldsymbol{\mu}(\boldsymbol{\xi}) \\ (1.11) \quad &= \int_{\mathbb{R}^n} i \frac{\xi_j}{|\boldsymbol{\xi}|} \mathbf{E}(\mathbf{x}, i\boldsymbol{\xi}) \mathcal{F}f(\boldsymbol{\xi}) d\boldsymbol{\mu}(\boldsymbol{\xi}). \end{aligned}$$

Our main result is the following theorem which is an analogue of the result about the characterization of the classical Hardy spaces by the classical Riesz transforms $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$ (see, e.g., [20, Chapter III, Section 4]).

Theorem 1.12. *Let $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$. Then $f \in H_{\max, \mathbf{L}}^1$ if and only if $\mathcal{R}_j f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ for $j = 1, 2, \dots, n$. Moreover, there exists a constant $C > 0$ such that*

$$(1.13) \quad C^{-1} \|f\|_{H_{\max, \mathbf{L}}^1} \leq \|f\|_{L^1(\mathbb{R}^n, d\boldsymbol{\mu})} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\boldsymbol{\mu})} \leq C \|f\|_{H_{\max, \mathbf{L}}^1}.$$

Let us emphasize that Theorem 1.12 implies a Riesz transform characterization of the Hardy space $H_{\max, \mathbb{L}}^1$ associated with multidimensional Bessel operator \mathbb{L} . To be more precise, on $(0, \infty)^n$ equipped with the measure $d\boldsymbol{\mu}$ we consider the Bessel operator

$$\mathbb{L} = \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} \right)$$

and the associated semigroup $\{e^{t\mathbb{L}}\}_{t>0}$. The action of the semigroup $e^{t\mathbb{L}}$ on functions is given by integration against the heat kernel $\mathbb{H}_t(\mathbf{x}, \mathbf{y})$, namely,

$$(1.14) \quad e^{t\mathbb{L}} f(\mathbf{x}) = \int_{(0, \infty)^n} \mathbb{H}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\boldsymbol{\mu}(\mathbf{y}),$$

where $\mathbb{H}_t(x, y) = \prod_{j=1}^n \mathbf{h}_t^{[j]}(x_j, y_j)$,

$$(1.15) \quad \mathbf{h}_t^{[j]}(x_j, y_j) = (2t)^{-1} \exp(-(x_j^2 + y_j^2)/4t) I_{k_j-1/2} \left(\frac{x_j y_j}{2t} \right) (x_j y_j)^{-k_j+1/2},$$

(see [24]). We define the Hardy space (see, e.g., [9])

$$H_{\max, \mathbb{L}}^1 = \left\{ f \in L^1((0, \infty)^n, d\boldsymbol{\mu}) : \left\| \sup_{t>0} |e^{t\mathbb{L}} f| \right\|_{L^1((0, \infty)^n, d\boldsymbol{\mu})} = \|f\|_{H_{\max, \mathbb{L}}^1} < \infty \right\}.$$

Let $\mathcal{R}_j = \partial_{x_j} \mathbb{L}^{-1/2}$ denote the Riesz transform associated with \mathbb{L} . Now we state our second main result.

Theorem 1.16. *Assume that $f \in L^1((0, \infty)^n, d\boldsymbol{\mu})$. Then f belongs to $H_{\max, \mathbb{L}}^1$ if and only if $R_j f \in L^1((0, \infty)^n, d\boldsymbol{\mu})$ for $j = 1, 2, \dots, n$. Moreover, there is a constant $C > 0$ such that*

$$(1.17) \quad C^{-1} \|f\|_{H_{\max, \mathbb{L}}^1} \leq \|f\|_{L^1((0, \infty)^n, d\boldsymbol{\mu})} + \sum_{j=1}^n \|R_j f\|_{L^1((0, \infty)^n, d\boldsymbol{\mu})} \leq C \|f\|_{H_{\max, \mathbb{L}}^1}.$$

2. POISSON SEMIGROUP

The Poisson semigroup $\{P_t\}_{t>0}$ in the Dunkl setting is defined by:

$$P_t f(\mathbf{x}) = e^{-t\sqrt{-\mathbf{L}}} f(\mathbf{x}) = \mathcal{F}^{-1}(e^{-t|\boldsymbol{\xi}|} \mathcal{F} f(\boldsymbol{\xi}))(\mathbf{x}) = \int_{\mathbb{R}^n} P_t(\mathbf{x}, \mathbf{y}) d\boldsymbol{\mu}(\mathbf{y}),$$

where the associated Poisson kernel is given by

$$P_t(\mathbf{x}, \mathbf{y}) = c \int_{\mathbb{R}^n} \mathbf{E}(\mathbf{x}, i\boldsymbol{\xi}) e^{-t|\boldsymbol{\xi}|} \mathbf{E}(\mathbf{y}, -i\boldsymbol{\xi}) d\boldsymbol{\mu}(\boldsymbol{\xi}).$$

By the subordination formula

$$(2.1) \quad P_t(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \mathbf{h}_{t^2/4u}(\mathbf{x}, \mathbf{y}) \frac{du}{\sqrt{u}}.$$

It easily follows from (2.1), (1.6), (1.2) and (1.4) that $P_t(\mathbf{x}, \mathbf{y}) = P_t(\mathbf{y}, \mathbf{x})$ is a positive smooth function of the $(t, \mathbf{x}, \mathbf{y})$ variables. We shall also show (see Appendix) that for every $1 \leq p < \infty$ and $t > 0$ there is a constant $C_{p,t}$ such that

$$(2.2) \quad \sup_{\mathbf{x} \in \mathbb{R}^n} \int_{\mathbb{R}^n} P_t(\mathbf{x}, \mathbf{y})^p d\boldsymbol{\mu}(\mathbf{y}) \leq C_{p,t}.$$

Let $\mathcal{L} = \frac{\partial^2}{\partial t^2} + \mathbf{L}$. Then

$$(2.3) \quad \mathcal{L} P_t f(\mathbf{x}) = 0.$$

Let

$$P_* f(\mathbf{x}) = \sup_{t>0} |P_t f(\mathbf{x})|$$

be the maximal operator associated with $\{P_t\}_{t>0}$.

In order to prove Theorem 1.12 we shall use Theorem 2.4 and Proposition 2.5. The proofs of them together with basic properties of $P_t(\mathbf{x}, \mathbf{y})$ are presented in the appendix.

Theorem 2.4. (a) *Let $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$. Then $f \in H_{\max, \mathbb{L}}^1$ if and only if the maximal function $P_* f$ belongs to $L^1(\mathbb{R}^n, d\boldsymbol{\mu})$. Moreover,*

$$\|P_* f\|_{L^1(\mathbb{R}^n, d\boldsymbol{\mu})} \sim \|f\|_{H_{\max, \mathbb{L}}^1}.$$

(b) *For every $1 < p \leq \infty$ the maximal operator P_* is bounded on $L^p(\mathbb{R}^n, d\boldsymbol{\mu})$.*

Proposition 2.5. (a) *Assume that $g \in L_{\text{loc}}^1(\mathbb{R}^n, d\boldsymbol{\mu})$ and $\lim_{|\mathbf{x}| \rightarrow \infty} |g(\mathbf{x})| = 0$. Then*

$$\lim_{(|\mathbf{x}|+t) \rightarrow \infty} P_t g(\mathbf{x}) = 0.$$

(b) *If $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$, then for every $\varepsilon > 0$ we have*

$$(2.6) \quad \lim_{(|\mathbf{x}|+t) \rightarrow \infty} P_{t+\varepsilon} f(\mathbf{x}) = 0.$$

3. KEY LEMMA

The following lemma, which is perhaps interesting in its own, will play a crucial role in the proof of Theorem 1.12.

Lemma 3.1. *For every positive integer n and every $\varepsilon > 0$ there is $\delta > 0$ such that for any matrix*

$$B = \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n} \\ b_{1,0} & b_{1,1} & \dots & b_{1,n} \\ & & \dots & \\ b_{n,0} & b_{n,1} & \dots & b_{n,n} \end{bmatrix}$$

with real entries we have

$$(3.2) \quad \|B\|^2 \leq (1 - \delta)\|B\|_{\text{HS}}^2 + \varepsilon \left((\text{tr } B)^2 + \sum_{i < j} (b_{i,j} - b_{j,i})^2 \right).$$

Here $\|B\| = \sup_{\mathbf{x} \in \mathbb{R}^{n+1}, \|\mathbf{x}\|=1} \|B\mathbf{x}\|$ is the ordinary norm of B and $\|B\|_{\text{HS}} = (\sum_{j=0}^n \sum_{\ell=0}^n b_{j,\ell}^2)^{1/2}$ is the Hilbert-Schmidt norm.

Proof. Let $S = (s_{i,j})_{i,j=0,1,\dots,n}$ and $A = (a_{i,j})_{i,j=0,1,\dots,n}$ denote any symmetric and antisymmetric matrix respectively. It is clear that

$$(3.3) \quad \|A + S\|_{\text{HS}}^2 = \|A\|_{\text{HS}}^2 + \|S\|_{\text{HS}}^2,$$

$$(3.4) \quad \sum_{i < j} (a_{i,j} - a_{j,i})^2 = 2\|A\|_{\text{HS}}^2.$$

It is known (see e.g., Section 3.1.2 of Chapter VII of [21]) that (3.2) holds for symmetric trace zero matrixes. Observe that it also holds for antisymmetric matrixes with $\delta = 2\varepsilon$. Indeed, from (3.4), we get

$$\|A\|^2 \leq \|A\|_{\text{HS}}^2 = (1 - 2\varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon \sum_{i < j} (a_{i,j} - a_{j,i})^2.$$

We claim that for any fixed $\varepsilon > 0$ and A, S such that $\|A\|_{\text{HS}}^2 \geq \frac{1}{\varepsilon}$, $\|S\|_{\text{HS}}^2 = 1$, we have

$$(3.5) \quad \|A + S\|^2 \leq (1 - \varepsilon)\|A + S\|_{\text{HS}}^2 + 2\varepsilon\|A\|_{\text{HS}}^2.$$

To see (3.5) we utilize (3.3) and obtain

$$\begin{aligned} \|A + S\|^2 &\leq \|A + S\|_{\text{HS}}^2 = \|A\|_{\text{HS}}^2 + \|S\|_{\text{HS}}^2 \\ &= (1 - \varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2 + 1 \\ &\leq (1 - \varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2 \\ &\leq (1 - \varepsilon)\|A + S\|_{\text{HS}}^2 + 2\varepsilon\|A\|_{\text{HS}}^2. \end{aligned}$$

Since (3.2) is homogeneous of degree 2, that is, $\|tB\|^2 = t^2\|B\|^2$, and

$$(1 - \delta)\|tB\|_{\text{HS}}^2 + \varepsilon \left((\text{tr } tB)^2 + \sum_{i < j} (tb_{i,j} - tb_{j,i})^2 \right) = t^2 \left((1 - \delta)\|B\|_{\text{HS}}^2 + \varepsilon \left((\text{tr } B)^2 + \sum_{i < j} (b_{i,j} - b_{j,i})^2 \right) \right),$$

it suffices to prove (3.2) for $B = A + S$ with S running over the unit sphere in the Hilbert-Schmidt norm, that is, $\|S\|_{\text{HS}}^2 = 1$. Assume that (3.2) does not hold. Then there is $\varepsilon > 0$ such that for every $\delta_n = \frac{1}{n}$ there are A_n and S_n , $\|S_n\|_{\text{HS}}^2 = 1$, such that

$$(3.6) \quad \|A_n + S_n\|^2 > (1 - \delta_n)\|A_n + S_n\|_{\text{HS}}^2 + 2\varepsilon\|A_n\|_{\text{HS}}^2 + \varepsilon(\text{tr } S_n)^2.$$

It follows from (3.5) that $\|A_n\|_{\text{HS}}^2 \leq \frac{1}{\varepsilon}$ for large n . Thus S_n and A_n are in compact sets. There is a subsequence n_k such that S_{n_k} and A_{n_k} converge to S and A respectively. Moreover, $\|S\|_{\text{HS}}^2 = 1$. Passing to limit in (3.6) as $k \rightarrow \infty$, we obtain

$$(3.7) \quad \|A + S\|^2 \geq \|A + S\|_{\text{HS}}^2 + 2\varepsilon\|A\|_{\text{HS}}^2 + \varepsilon(\text{tr } S)^2.$$

The inequality (3.7) implies that $A = 0$ and $\text{tr } S = 0$. Hence $\|S\|^2 \geq \|S\|_{\text{HS}}^2$, which is impossible for a nonzero symmetric matrix S with $\text{tr } S = 0$. \square

4. RIESZ TRANSFORMS AND CAUCHY-RIEMANN EQUATIONS

For a function $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ such that $\mathcal{R}_j f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ we define the functions

$$(4.1) \quad \begin{aligned} u_0(t, \mathbf{x}) &= P_t f(\mathbf{x}) = \int_{\mathbb{R}^n} e^{-t|\boldsymbol{\xi}|} \mathcal{F}f(\boldsymbol{\xi}) \mathbf{E}(\mathbf{x}, i\boldsymbol{\xi}) d\boldsymbol{\mu}(\boldsymbol{\xi}), \\ u_j(t, \mathbf{x}) &= -P_t(\mathcal{R}_j f)(\mathbf{x}) = - \int_{\mathbb{R}^n} i \frac{\xi_j}{|\boldsymbol{\xi}|} e^{-t|\boldsymbol{\xi}|} \mathcal{F}f(\boldsymbol{\xi}) \mathbf{E}(\mathbf{x}, i\boldsymbol{\xi}) d\boldsymbol{\mu}(\boldsymbol{\xi}). \end{aligned}$$

The functions u_j , $j = 0, 1, \dots, n$, are C^∞ on $(0, \infty) \times \mathbb{R}^n$. It is easy to check using (1.3) and (4.1) that they satisfy the following Cauchy-Riemann type equations:

$$(4.2) \quad \begin{aligned} D_j u_0(t, \mathbf{x}) &= \partial_t u_j(t, \mathbf{x}), \quad j = 1, \dots, n; \\ D_j u_\ell(t, \mathbf{x}) &= D_\ell u_j(t, \mathbf{x}), \quad j, \ell = 1, \dots, n; \\ \partial_t u_0(t, \mathbf{x}) + \sum_{j=1}^n D_j u_j(t, \mathbf{x}) &= 0. \end{aligned}$$

From now we shall assume that f is real-valued, then so are u_j , $j = 0, 1, \dots, n$.

Let \mathcal{G} denote the group of reflections in \mathbb{R}^n generated by σ_j , $j = 1, \dots, n$. For $\sigma \in \mathcal{G}$ and a function $u(t, \mathbf{x})$ defined on $(0, \infty) \times \mathbb{R}^n$ we denote $u^\sigma(t, \mathbf{x}) = u(t, \sigma\mathbf{x})$. If $\mathbf{u}(t, \mathbf{x}) = (u_0(t, \mathbf{x}), u_1(t, \mathbf{x}), \dots, u_n(t, \mathbf{x}))$ satisfies (4.2), then so does $\mathbf{u}^\sigma(t, \mathbf{x}) = (u_0^\sigma(t, \mathbf{x}), u_1^\sigma(t, \mathbf{x}), \dots, u_n^\sigma(t, \mathbf{x}))$.

Moreover, if $(u_0(t, \mathbf{x}), u_1(t, \mathbf{x}), \dots, u_n(t, \mathbf{x}))$ is of the form (4.1), then

$$u_0^\sigma(t, \mathbf{x}) = P_t(f^\sigma)(\mathbf{x}), \quad u_j^\sigma(t, \mathbf{x}) = P_t(\mathcal{R}_j(f^\sigma))(\mathbf{x}),$$

where $f^\sigma(\mathbf{x}) = f(\sigma\mathbf{x})$.

For a C^2 function $\mathbf{u}(t, \mathbf{x}) = (u_0(t, \mathbf{x}), u_1(t, \mathbf{x}), \dots, u_n(t, \mathbf{x}))$ satisfying (4.2) consider the function $F : (0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^{(n+1) \cdot |\mathcal{G}|}$,

$$F(t, \mathbf{x}) = \{\mathbf{u}^\sigma(t, \mathbf{x})\}_{\sigma \in \mathcal{G}}.$$

Observe that $|F(t, \mathbf{x})| = |F(t, \sigma\mathbf{x})|$ for every $\sigma \in \mathcal{G}$, where

$$|F(t, \mathbf{x})|^2 = \sum_{\sigma \in \mathcal{G}} \sum_{\ell=0}^n |u_\ell^\sigma(t, \mathbf{x})|^2.$$

Our main task is to prove that the following proposition, which is an analogue of the classical result (see, e.g., [21, Section 3.1 of Chapter VII]).

Proposition 4.3. *There is an exponent $0 < q < 1$ which depends on k_1, \dots, k_n such that the function $|F|^q$ is \mathcal{L} -subharmonic, that is, $\mathcal{L}(|F|^q)(t, \mathbf{x}) \geq 0$ on the set where $|F| > 0$.*

Proof. Observe that $|F|^q$ is C^2 on the set where $|F| > 0$. Let \cdot denote the inner product in $\mathbb{R}^{(n+1) \cdot |\mathcal{G}|}$. In order to unify our notation we denote the variable t by x_0 . For $j = 0, 1, \dots, n$,

we have

$$\begin{aligned}\partial_{x_j}|F|^q &= q|F|^{q-2}\left((\partial_{x_j}F) \cdot F\right) \\ \partial_{x_j}^2|F|^q &= q(q-2)|F|^{q-4}\left((\partial_{x_j}F) \cdot F\right)^2 + q|F|^{q-2}\left((\partial_{x_j}^2F) \cdot F + |\partial_{x_j}F|^2\right).\end{aligned}$$

Recall that $|F(x_0, \mathbf{x})| = |F(x_0, \sigma\mathbf{x})|$. Hence,

$$\begin{aligned}(4.4) \quad \mathcal{L}|F|^q &= q(q-2)|F|^{q-4}\left\{\left((\partial_{x_0}F) \cdot F\right)^2 + \sum_{j=1}^n\left((\partial_{x_j}F) \cdot F\right)^2\right\} \\ &\quad + q|F|^{q-2}\left\{\left(\partial_{x_0}^2F + \sum_{j=1}^n\left(\partial_{x_j}^2F + \frac{2k_j}{x_j}(\partial_{x_j}F)\right)\right) \cdot F + |\partial_{x_0}F|^2 + \sum_{j=1}^n|\partial_{x_j}F|^2\right\}.\end{aligned}$$

Since $D_j D_\ell f = D_\ell D_j f$ for $f \in C^2(\mathbb{R}^n)$, we conclude from (4.2) that for $\ell = 0, 1, \dots, n$ and $\sigma \in \mathcal{G}$ we have

$$\partial_{x_0}^2 u_\ell^\sigma + \sum_{j=1}^n \left(\partial_{x_j}^2 u_\ell^\sigma + \frac{2k_j}{x_j} (\partial_{x_j} u_\ell^\sigma) \right) = \sum_{j=1}^n \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j}).$$

Thus,

$$\begin{aligned}(4.5) \quad \left(\partial_{x_0}^2 F + \sum_{j=1}^n \left(\partial_{x_j}^2 F + \frac{2k_j}{x_j} (\partial_{x_j} F) \right) \right) \cdot F &= \sum_{\sigma \in \mathcal{G}} \sum_{\ell=0}^n \sum_{j=1}^n \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j}) u_\ell^\sigma \\ &= \sum_{j=1}^n \sum_{\ell=0}^n \sum_{\sigma \in \mathcal{G}} \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j}) u_\ell^\sigma \\ &= \frac{1}{2} \sum_{j=1}^n \sum_{\ell=0}^n \sum_{\sigma \in \mathcal{G}} \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j})^2 \\ &= \frac{1}{2} \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \sum_{\ell=0}^n \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j})^2.\end{aligned}$$

Thanks to (4.4) and (4.5), it suffices to prove that there is $0 < q < 1$ such that

$$\begin{aligned}(4.6) \quad (2-q) \left\{ \left((\partial_{x_0}F) \cdot F \right)^2 + \sum_{j=1}^n \left((\partial_{x_j}F) \cdot F \right)^2 \right\} \\ \leq \frac{1}{2} |F|^2 \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \sum_{\ell=0}^n \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j})^2 + |F|^2 \left(|\partial_{x_0}F|^2 + \sum_{j=1}^n |\partial_{x_j}F|^2 \right).\end{aligned}$$

Denote

$$B_\sigma = \begin{bmatrix} \partial_{x_0} u_0^\sigma & \partial_{x_0} u_1^\sigma & \dots & \partial_{x_0} u_n^\sigma \\ \partial_{x_1} u_0^\sigma & \partial_{x_1} u_1^\sigma & \dots & \partial_{x_1} u_n^\sigma \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_n} u_0^\sigma & \partial_{x_n} u_1^\sigma & \dots & \partial_{x_n} u_n^\sigma \end{bmatrix}.$$

Let $\mathbf{B} = \{B_\sigma\}_{\sigma \in \mathcal{G}}$ be matrix with $n+1$ rows and $(n+1) \cdot |\mathcal{G}|$ columns. It represents a linear operator from $\mathbb{R}^{(n+1) \cdot |\mathcal{G}|}$ into \mathbb{R}^{n+1} .

Observe that

$$(2-q) \left\{ \left((\partial_{x_0}F) \cdot F \right)^2 + \sum_{j=1}^n \left((\partial_{x_j}F) \cdot F \right)^2 \right\} \leq (2-q) |F|^2 \|\mathbf{B}\|^2,$$

$$|F|^2 \left(|\partial_{x_0} F|^2 + \sum_{j=1}^n |\partial_{x_j} F|^2 \right) = |F|^2 \|\mathbf{B}\|_{\text{HS}}^2.$$

Clearly,

$$\|\mathbf{B}\|^2 \leq \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2, \quad \|\mathbf{B}\|_{\text{HS}}^2 = \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|_{\text{HS}}^2.$$

Therefore the inequality (4.6) will be proven if we show that

$$(4.7) \quad (2-q) \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2 \leq \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|_{\text{HS}}^2 + \frac{1}{2} \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \sum_{\ell=0}^n \frac{k_j}{x_j^2} (u_\ell^\sigma - u_\ell^{\sigma\sigma_j})^2.$$

Applying the Cauchy-Riemann type equations (4.2), we obtain

$$(4.8) \quad (\text{tr} B_\sigma)^2 = \left(- \sum_{j=1}^n \frac{k_j}{x_j} (u_j^\sigma - u_j^{\sigma\sigma_j}) \right)^2 \leq \left(\sum_{s=1}^n k_s \right) \left(\sum_{j=1}^n \frac{k_j}{x_j^2} (u_j^\sigma - u_j^{\sigma\sigma_j})^2 \right),$$

$$(4.9) \quad \begin{aligned} \sum_{i < j} (\partial_{x_i} u_j^\sigma - \partial_{x_j} u_i^\sigma)^2 &= \sum_{j=1}^n \frac{k_j^2}{x_j^2} (u_0^\sigma - u_0^{\sigma\sigma_j})^2 + \sum_{1 \leq i < j} \left(\frac{k_i}{x_i} (u_j^\sigma - u_j^{\sigma\sigma_i}) - \frac{k_j}{x_j} (u_i^\sigma - u_i^{\sigma\sigma_j}) \right)^2 \\ &\leq 2 \left(\sum_{s=1}^n k_s \right) \left(\sum_{i=0}^n \sum_{j=1}^n \frac{k_j}{x_j^2} (u_i^\sigma - u_i^{\sigma\sigma_j})^2 \right). \end{aligned}$$

Using Lemma 3.1 together with (4.8) and (4.9) we have that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$(4.10) \quad \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2 \leq (1-\delta) \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|_{\text{HS}}^2 + 3\varepsilon \left(\sum_{s=1}^n k_s \right) \sum_{\sigma \in \mathcal{G}} \left(\sum_{i=0}^n \sum_{j=1}^n \frac{k_j}{x_j^2} (u_i^\sigma - u_i^{\sigma\sigma_j})^2 \right).$$

Taking $\varepsilon > 0$ such that $3\varepsilon \sum_{s=1}^n k_s \leq \frac{1}{4}$ and utilizing (4.10) we deduce that (4.7) holds with $(1-\delta) \leq (2-q)^{-1}$. \square

5. MAXIMUM PRINCIPLE

On $\mathbb{R}_+ \times \mathbb{R}^n$ let

$$\tilde{\mathbb{L}} = \frac{\partial^2}{\partial x_0^2} + \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} \right).$$

Proposition 5.1. *Let $f(x_0, x_1, \dots, x_n)$ be a C^2 function defined on an open connected set $\Omega \subset (0, \infty) \times \mathbb{R}^n$. Assume that $f(x_0, x_1, \dots, -x_j, \dots, x_n) = f(x_0, x_1, \dots, x_j, \dots, x_n)$ whenever $(x_0, x_1, \dots, -x_j, \dots, x_n)$ and $(x_0, x_1, \dots, x_j, \dots, x_n)$ belong to Ω and $\tilde{\mathbb{L}}f \geq 0$ on the set $\{(x_0, \dots, x_n) \in \Omega : x_1 \cdot x_2 \cdot \dots \cdot x_n \neq 0\}$. Then f cannot attain a local maximum in Ω unless f is a constant.*

Proof. The proposition is a corollary of Theorem 4.2 of [17]. For the convenience of the reader we present here an alternative proof based on ideas from [14], where the one dimensional Bessel operator was considered.

Denote $\mathbf{x} = (x_0, \mathbf{x}) = (x_0, x_1, \dots, x_n) \in (0, \infty) \times \mathbb{R}^n$, $U = \{\mathbf{x} \in (0, \infty) \times \mathbb{R}^n : x_1 \cdot x_2 \cdot \dots \cdot x_n \neq 0\}$. Set

$$v(x_0, x_1, \dots, x_n) = |x_1|^{2k_1} |x_2|^{2k_2} \cdot \dots \cdot |x_n|^{2k_n}.$$

By the divergence theorem for C^2 functions f and g in a smooth region \bar{D} one has

$$(5.2) \quad \int_D [g \operatorname{div}(v \nabla f) - f \operatorname{div}(v \nabla g)] dx_0 dx_1 \dots dx_n = \int_{\partial D} v \left(g \frac{\partial f}{\partial \mathbf{n}} - f \frac{\partial g}{\partial \mathbf{n}} \right) ds,$$

where \mathbf{n} is outward normal vector to D at $\mathbf{x} \in \partial D$. Observe that $\operatorname{div}(v\nabla f) = v\tilde{\mathbb{L}}f$ on $D \cap U$. So, if g is additionally $\tilde{\mathbb{L}}$ -harmonic on $D \cap U$ then $\operatorname{div}(v\nabla g) = v\tilde{\mathbb{L}}g = 0$ on $D \cap U$, and setting $f \equiv 1$ in (5.2) we get

$$(5.3) \quad \int_{\partial D} v \frac{\partial g}{\partial \mathbf{n}} ds = 0.$$

Assume that at $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \Omega$ the function f attains a local maximum. By Hopf's maximum principle (see [10, Section 6.4.2, Theorem 3]) \mathbf{a} is not a regular point of $\tilde{\mathbb{L}}$, that is, $a_1 \cdot a_2 \cdot \dots \cdot a_n = 0$. There is no loss of generality in assuming that there is $m \in \{1, 2, \dots, n\}$ such that $a_0 > 0, \dots, a_{m-1} > 0, a_m = a_{m+1} = \dots = a_n = 0$. Let

$$\begin{aligned} h_\tau^{[0]}(x_0, a_0) &= \frac{1}{\sqrt{4\pi\tau}} \exp(-|x_0 - a_0|^2/4\tau), \\ h_\tau^{[j]}(x_j, 0) &= \frac{1}{\Gamma(\lambda_j + 1/2)} \tau^{-k_j-1/2} \exp(-x_j^2/4\tau), \quad j \in \{m, m+1, \dots, n\}, \\ h_\tau^{[j]}(x_j, a_j) &= h_\tau^{[j]}(x_j, a_j) \quad \text{for } j \notin \{0, m, m+1, \dots, n\} \end{aligned}$$

(see (1.15)). Put

$$g_0(x_0, x_1, \dots, x_n) = \int_0^\infty \prod_{j=0}^n h_\tau^{[j]}(x_j, a_j) d\tau.$$

We have $\tilde{\mathbb{L}}g_0 = 0$ on $((0, \infty)^m \times \mathbb{R}^{n+1-m}) \cap U$. It is not difficult to check using the asymptotic behavior of the Bessel functions I_ν (see, e.g., [15]) that there is $r > 0$ such that $\nabla g_0(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in B(\mathbf{a}, 2r) \setminus \{\mathbf{a}\} \subset D$. Let $D_R = \{\mathbf{x} : g_0(\mathbf{x}) > R\} \cup \{\mathbf{a}\}$. We take R large enough such that $D_R \subset B(\mathbf{a}, r)$. For $\varepsilon > 0$ small enough let $D_{R,\varepsilon} = D_R \setminus B(\mathbf{a}, \varepsilon)$. Set $g(\mathbf{x}) = g_0(\mathbf{x}) - R$. Then $g \equiv 0$ on ∂D_R , $g \geq 0$ on $D_{R,\varepsilon}$ and $\frac{\partial}{\partial \mathbf{n}} g(\mathbf{x}) < 0$ on ∂D_R , where \mathbf{n} is outward normal vector to D_R at $\mathbf{x} \in \partial D_R$. Using (5.3) we have

$$(5.4) \quad \int_{\partial D_R} v \frac{\partial g}{\partial \mathbf{n}} ds = \int_{\partial B(\mathbf{a}, \varepsilon)} v \frac{\partial g}{\partial \mathbf{n}} ds < 0.$$

Now from (5.2) we conclude that

$$\begin{aligned} (5.5) \quad 0 &\leq \int_{D_{R,\varepsilon}} g(v\tilde{\mathbb{L}}f) dx_0 dx_1 \dots dx_n \\ &= \int_{D_{R,\varepsilon}} g \operatorname{div}(v\nabla f) dx_0 dx_1 \dots dx_n \\ &= - \int_{\partial D_R} f v \frac{\partial g}{\partial \mathbf{n}} ds - \int_{\partial B(\mathbf{a}, \varepsilon)} g v \frac{\partial f}{\partial \mathbf{n}} ds + \int_{\partial B(\mathbf{a}, \varepsilon)} f v \frac{\partial g}{\partial \mathbf{n}} ds, \end{aligned}$$

where in the last two integrals \mathbf{n} is the outward normal vector to $B(\mathbf{a}, \varepsilon)$. Clearly, the second summand tends to 0 as ε tends to 0. On the other hand, by (5.3), the third summand tends to $f(\mathbf{a}) \int_{\partial D_R} v \frac{\partial g}{\partial \mathbf{n}} ds$. Thus,

$$(5.6) \quad 0 \leq \int_{\partial D_R} (f(\mathbf{a}) - f) v \frac{\partial g}{\partial \mathbf{n}} ds.$$

Recall that f attains a local maximum at \mathbf{a} and $\frac{\partial g}{\partial \mathbf{n}} < 0$ on ∂D_R . Hence, from (5.6) we deduce that $f = f(\mathbf{a})$ on ∂D_R . So f must be a constant in a neighborhood of \mathbf{a} and, consequently, $f \equiv f(\mathbf{a})$ on Ω , since Ω is connected. \square

6. PROOF OF THEOREM 1.12

Proof of Theorem 1.12. The second inequality in (1.13) is a direct consequence of the following multiplier theorem (see [1, Theorem 1.10]).

Theorem 6.1. *Let $\chi = \chi(\xi)$ be a smooth radial function on \mathbb{R}^n such that*

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \in [\frac{1}{2}, 2], \\ 0 & \text{if } |\xi| \notin (\frac{1}{4}, 4). \end{cases}$$

If a function $m = m(\xi)$ on \mathbb{R}^n satisfies

$$(6.2) \quad M = \sup_{t>0} \|\chi m(t \cdot)\|_{W_2^{\mathbf{N}/2+\varepsilon}} < +\infty,$$

for some $\varepsilon > 0$, then the multiplier operator

$$\mathcal{T}_m f = \mathcal{F}^{-1}\{m(\mathcal{F}f)\}$$

is bounded on the Hardy space $H_{\max, \mathbf{L}}^1$ and

$$\|\mathcal{T}_m\|_{H_{\max, \mathbf{L}}^1 \rightarrow H_{\max, \mathbf{L}}^1} \lesssim M.$$

It is not difficult to check that the multiplier $m_j(\xi) = i \frac{\xi_j}{|\xi|}$, which corresponds to the Riesz transform \mathcal{R}_j , satisfies (6.2). Hence \mathcal{R}_j is bounded from $H_{\max, \mathbf{L}}^1$ to itself and, consequently, from $H_{\max, \mathbf{L}}^1$ to $L^1(\mathbb{R}^n, d\mu)$.

Now we turn to prove the first inequality in (1.13). For this purpose we use Theorem 2.4, Propositions 2.5, 4.3, and 5.1 combined with the steps of the proof of the characterization of the classical Hardy spaces by the classical Riesz transforms (see, e.g., [20, Chapter III, Section 4]). For the convenience of the reader, we provide the details.

Assume that $f \in L^1(\mathbb{R}^n, d\mu)$ and $\mathcal{R}_j f \in L^1(\mathbb{R}^n, d\mu)$ for $j = 1, 2, \dots, n$. There is no loss of generality in assuming that f is real valued, and hence so are $\mathcal{R}_j f$. Set $\mathbf{u}(x_0, x_1, \dots, x_n) = (u_0, u_1, \dots, u_n)$, where u_j are defined in (4.1) (recall that $x_0 = t > 0$). Fix $0 < q < 1$ as in Proposition 4.3 and set $p = 1/q$. Let $F(x_0, \mathbf{x}) = \{\mathbf{u}^\sigma(x_0, \mathbf{x})\}_{\sigma \in \mathcal{G}}$. Clearly,

$$(6.3) \quad \sup_{x_0 > 0} \int_{\mathbb{R}^n} |F(x_0, x_1, \dots, x_n)| d\mu(x_1, \dots, x_n) \leq C \left(\|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right).$$

Denote $F_\varepsilon(x_1, \dots, x_n) = F(\varepsilon, x_1, \dots, x_n)$. Then $|F_\varepsilon| \in C_0(\mathbb{R}^n)$ (see part (b) of Proposition 2.5) and, by (8.6),

$$(6.4) \quad \sup_{\varepsilon > 0} \| |F_\varepsilon|^q \|_{L^p(\mathbb{R}^n, d\mu)}^p \leq C \left(\|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right).$$

Consider the function $G : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$G(x_0, x_1, \dots, x_n) = |F(\varepsilon + x_0, x_1, \dots, x_n)|^q - P_{x_0}(|F_\varepsilon|^q)(x_1, \dots, x_n).$$

The function is continuous vanishes for $x_0 = 0$ and, by Proposition 2.5,

$$\lim_{(x_0 + |(x_1, \dots, x_n)|) \rightarrow \infty} G(x_0, x_1, \dots, x_n) = 0.$$

Moreover, $G(x_0, \mathbf{x}) = G(x_0, \sigma \mathbf{x})$ for every $\sigma \in \mathcal{G}$. We claim that

$$(6.5) \quad G(x_0, \mathbf{x}) = |F(\varepsilon + x_0, \mathbf{x})|^q - P_{x_0}(|F_\varepsilon|^q)(\mathbf{x}) \leq 0.$$

To prove the claim assume that $G > 0$ at some point. Then it attains a global maximum, say at $\mathbf{a} = (a_0, a_1, \dots, a_n)$. Obviously, $a_0 > 0$ and $|F(\mathbf{a})| > 0$. Take a connected neighborhood Ω of \mathbf{a} such that $G > 0$ on Ω and G is not constant on Ω . Then G is C^2 on Ω and, according to

(2.3) and Proposition 4.3, $\mathcal{L}G = \tilde{\mathbb{L}}G = \tilde{\mathbb{L}}|F|^q \geq 0$ on $\{(x_0, x_1, \dots, x_n) \in \Omega : x_1 \cdot x_2 \cdot \dots \cdot x_n \neq 0\}$. This contradicts the maximum principle (see Proposition 5.1). Hence (6.5) is proved.

It follows from (6.4) that there is a sequence $\varepsilon_n \rightarrow 0$ such that $|F_{\varepsilon_n}|^q$ converges in a weak $*$ topology of the Banach space $L^p(\mathbb{R}^n, d\mu)$ to $h \in L^p(\mathbb{R}^n, d\mu)$ and

$$(6.6) \quad \|h\|_{L^p(\mathbb{R}^n, d\mu)}^p \leq C \left(\|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right).$$

From (6.5) and (2.2) we conclude that

$$(6.7) \quad |F(x_0, \mathbf{x})|^q \leq P_{x_0} h(\mathbf{x}).$$

Since the maximal function P_* is bounded on $L^p(\mathbb{R}^n, d\mu(x))$ (see Theorem 2.4), we deduce from (6.7) and (6.6) that

$$\begin{aligned} \int \sup_{x_0 > 0} |u_0(x_0, \mathbf{x})| d\mu(\mathbf{x}) &\leq C \int \left(\sup_{x_0 > 0} P_{x_0} h(\mathbf{x}) \right)^p d\mu(\mathbf{x}) \\ &\leq C \|h\|_{L^p(\mathbb{R}^n, d\mu)}^p \\ &\leq C \left(\|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right). \end{aligned}$$

Finally, from Theorem 2.4 we get $f \in H_{\max, \mathbf{L}}^1$ and

$$\|f\|_{H_{\max, \mathbf{L}}^1} \leq C \left(\|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right).$$

□

7. PROOF OF THEOREM 1.16

Proof of Theorem 1.16. Recall that

$$(7.1) \quad \mathbf{h}_t(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^n h_t^{\{j\}}(x_j, y_j)$$

(see (1.2) and (1.4)), where

$$h_t^{\{j\}}(x, y) = (4t)^{-1} \exp(-(x^2 + y^2)/4t) |xy|^{-k_j+1/2} \left(I_{k_j-1/2} \left(\frac{|xy|}{2t} \right) + \operatorname{sgn}(xy) I_{k_j+1/2} \left(\frac{|xy|}{2t} \right) \right)$$

is the heat kernel associated with one dimensional Dunkl operator

$$Lf(x) = f''(x) + \frac{2k_j}{x} f'(x) - \frac{k_j}{x^2} (f(x) - f(-x)).$$

Clearly, $h_t^{\{j\}}(x, y) = h_t^{\{j\}}(y, x)$ is a C^∞ function of (t, x, y) . For a function f defined on $(0, \infty)^n$ let \tilde{f} denote its extension to the \mathcal{G} invariant function on \mathbb{R}^n . One can easily check using (1.14), (1.15) that

$$(e^{t\mathbb{L}} f)^\sim = e^{t\mathbf{L}}(\tilde{f}).$$

Hence, f belongs to the Hardy space $H_{\max, \mathbb{L}}^1$ if and only if $\tilde{f} \in H_{\max, \mathbf{L}}^1$. Moreover, $\|f\|_{H_{\max, \mathbb{L}}^1} = c \|\tilde{f}\|_{H_{\max, \mathbf{L}}^1}$. Let us note that

$$|\mathcal{R}_j \tilde{f}| = |(R_j f)^\sim|.$$

Thus Theorem 1.16 follows from Theorem 1.12. □

8. APPENDIX

It is well known that

$$(8.1) \quad \int_{\mathbb{R}} h_t^{\{j\}}(x, y) d\mu_j(y) = 1, \quad d\mu_j(y) = |y|^{2k_j} dy.$$

It was proved in [1] that $h_t^{\{j\}}(x, y)$ has the following global behavior :

$$(8.2) \quad h_t^{\{j\}}(x, y) \asymp \begin{cases} t^{-k_j - \frac{1}{2}} e^{-\frac{x^2 + y^2}{4t}} & \text{if } |xy| \leq t, \\ t^{-\frac{1}{2}} (xy)^{-k_j} e^{-\frac{(x-y)^2}{4t}} & \text{if } xy \geq t, \\ t^{\frac{1}{2}} (-xy)^{-k_j - 1} e^{-\frac{(x+y)^2}{4t}} & \text{if } -xy \geq t. \end{cases}$$

From (8.2) we easily conclude that

$$(8.3) \quad 0 < \mathbf{h}_t(\mathbf{x}, \mathbf{y}) \leq \frac{C}{\mu(B(\mathbf{x}, \sqrt{t}))} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } t > 0;$$

$$(8.4) \quad \mathbf{h}_t(\mathbf{x}, \mathbf{y}) \leq \frac{C}{\mu(B(\mathbf{x}, \sqrt{t}))} e^{-c|\mathbf{x}|^2/t} \quad \text{for } |\mathbf{x}| > 2n|\mathbf{y}|.$$

We shall need the following inequalities for volumes of the Euclidean balls (see [1])

$$(8.5) \quad \left(\frac{R}{r}\right)^n \lesssim \frac{\mu(B(\mathbf{x}, R))}{\mu(B(\mathbf{x}, r))} \lesssim \left(\frac{R}{r}\right)^N, \quad \forall \mathbf{x} \in \mathbb{R}^n, \forall R \geq r > 0.$$

The subordination formula (2.1) combined with (7.1) and (8.1) implies

$$(8.6) \quad \int_{\mathbb{R}^n} P_t(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{x}) = \int_{\mathbb{R}^n} P_t(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) = 1.$$

Lemma 8.7. *There is a constant $C > 0$ such that*

$$(8.8) \quad 0 < P_t(\mathbf{x}, \mathbf{y}) \leq \frac{C}{\mu(B(\mathbf{x}, t))}.$$

Moreover, for every $0 < \delta < \frac{1}{N}$ there is a constant C_δ such that

$$(8.9) \quad P_t(\mathbf{x}, \mathbf{y}) \leq \frac{C_\delta}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \quad \text{for } |\mathbf{x}| > 2n|\mathbf{y}|.$$

Proof. To see (8.8) we use (2.1) together with (8.3) and (8.5) and obtain

$$\begin{aligned} P_t(\mathbf{x}, \mathbf{y}) &\lesssim \int_0^{\frac{1}{4}} \frac{1}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}} + \int_{\frac{1}{4}}^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}} \\ &\lesssim \frac{1}{\mu(B(\mathbf{x}, t))} \left(\int_0^{\frac{1}{4}} u^{n/2} \frac{du}{\sqrt{u}} + \int_{\frac{1}{4}}^\infty e^{-u} u^{N/2} \frac{du}{\sqrt{u}} \right) \lesssim \frac{1}{\mu(B(\mathbf{x}, t))}. \end{aligned}$$

The proof of the lower bound of $P_t(\mathbf{x}, \mathbf{y})$ is obvious.

In order to prove (8.9) it suffices to consider $t \leq |\mathbf{x}|/2$. By (8.5), for every $\delta > 0$ and $c > 0$, we have

$$(8.10) \quad \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, \sqrt{s}))}\right)^{1+\delta} \leq C_\delta \left(1 + \frac{|\mathbf{x}|}{\sqrt{s}}\right)^{(1+\delta)N} \leq C_{\delta, c} e^{c|\mathbf{x}|^2/s}, \quad \text{for } s > 0.$$

Utilizing (8.4) together with (8.10) and proceeding similarly to the proof of (8.8) we have

$$\begin{aligned}
P_t(\mathbf{x}, \mathbf{y}) &\lesssim \int_0^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}} \\
&= \int_0^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}} \\
&\lesssim \int_0^{\frac{1}{4}} \frac{1}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \left(\frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}} \\
&\quad + \int_{\frac{1}{4}}^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \frac{du}{\sqrt{u}}.
\end{aligned}$$

Fix $0 < \delta < N^{-1}$. Applying (8.5) we obtain

$$\begin{aligned}
P_t(\mathbf{x}, \mathbf{y}) &\lesssim \int_0^{\frac{1}{4}} \frac{1}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} u^{-\delta N/2} \frac{du}{\sqrt{u}} \\
&\quad + \int_{\frac{1}{4}}^\infty \frac{e^{-u} u^{N/2}}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \frac{du}{\sqrt{u}},
\end{aligned}$$

which proves (8.9). \square

Proof of Theorem 2.4. The proof, which is in spirit similar to that of the heat kernel characterization of H_{atom}^1 (see [1]), is based on the following result due to Uchiyama [23].

Theorem 8.11. *Assume that a set X is equipped with*

- *a quasi-distance \tilde{d} i.e. a distance except that the triangular inequality is replaced by the weaker condition*

$$\tilde{d}(x, y) \leq A \{ \tilde{d}(x, z) + \tilde{d}(z, y) \}, \quad \forall x, y, z \in X;$$

- *a measure μ whose values on quasi-balls satisfy*

$$\frac{r}{A} \leq \mu(\tilde{B}(x, r)) \leq r, \quad \forall x \in X, \forall r > 0;$$

- *a continuous kernel $K_r(x, y) \geq 0$ such that, for every $r > 0$ and $x, y, y' \in X$,*

$$\begin{aligned}
&\circ K_r(x, x) \geq \frac{1}{Ar}, \\
&\circ K_r(x, y) \leq r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-\delta}, \\
&\circ |K_r(x, y) - K_r(x, y')| \leq r^{-1} \left(1 + \frac{\tilde{d}(x, y)}{r}\right)^{-1-2\delta} \left(\frac{\tilde{d}(y, y')}{r}\right)^\delta \text{ when } \tilde{d}(y, y') \leq \frac{r + \tilde{d}(x, y)}{4A}.
\end{aligned}$$

Here $A \geq 1$ and $\delta > 0$. Then the following definitions of the Hardy space $H^1(X)$ and their corresponding norms are equivalent:

- **Maximal definition:** $H_{\text{max}, K_r}^1(X)$ consists of all functions $f \in L^1(X, d\mu)$ such that

$$K_* f(x) = \sup_{r>0} \left| \int_X K_r(x, y) f(y) d\mu(y) \right|$$

belongs to $L^1(X, d\mu)$ and the norm $\|f\|_{H_{\text{max}, K_r}^1(X)} = \|K_* f\|_{L^1(X, d\mu)}$.

- **Atomic definition:** An atom for $H_{\text{atom}}^1(X, \tilde{d})$ is a measurable function $a : X \rightarrow \mathbb{C}$ such that: a is supported in a quasi-ball \tilde{B} , $\|a\|_{L^\infty} \lesssim \mu(\tilde{B})^{-1}$ and $\int_X a d\mu = 0$ (see, [4, 13, 23]). Then $H_{\text{atom}}^1(X, \tilde{d})$ consists of all functions $f \in L^1(X, d\mu)$ which can be written as $f = \sum_\ell \lambda_\ell a_\ell$, where the a_ℓ 's are atoms and $\sum_\ell |\lambda_\ell| < +\infty$, and the norm $\|f\|_{H_{\text{atom}}^1(X, \tilde{d})} = \inf \sum_\ell |\lambda_\ell|$ over all such representations.

For $X = \mathbb{R}^n$, equipped with the Euclidean distance $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ and the measure μ (see (1.5)), set

$$\tilde{d}(\mathbf{x}, \mathbf{y}) = \inf \mu(B), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

where the infimum is taken over all closed balls B containing \mathbf{x} and \mathbf{y} . Let $t = t(\mathbf{x}, r)$ be defined by $\mu(B(\mathbf{x}, \sqrt{t})) = r$. Then

$$\mu(\tilde{B}(\mathbf{x}, r)) \sim r$$

and there exists a constant $c > 0$ such that

$$(8.12) \quad B(\mathbf{x}, \sqrt{t}) \subset \tilde{B}(\mathbf{x}, r) \subset B(\mathbf{x}, c\sqrt{t}),$$

where $\tilde{B}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \tilde{d}(\mathbf{x}, \mathbf{y}) < r\}$ (see, e.g., [1]).

Let us remark that thanks to (8.12) and (8.5) the atomic spaces $H_{\text{atom}}^1(X, \tilde{d})$ and H_{atom}^1 (defined in Section 1) do coincide and $\|f\|_{H_{\text{atom}}^1(X, \tilde{d})} \sim \|f\|_{H_{\text{atom}}^1}$.

It was proved in [1] that the kernel \mathbf{h}_t can be written in the form

$$\mathbf{h}_t(\mathbf{x}, \mathbf{y}) = \mathbf{H}_t(\mathbf{x}, \mathbf{y}) + \mathbf{S}_t(\mathbf{x}, \mathbf{y}),$$

where $\mathbf{H}_t(\mathbf{x}, \mathbf{y})$ and $\mathbf{S}_t(\mathbf{x}, \mathbf{y})$ are nonnegative functions such that there are $C_1, C_2, C_4, \delta > 0$ such that

$$(8.13) \quad \mathbf{H}_t(\mathbf{x}, \mathbf{x}) \geq \frac{C_1}{\mu(B(\mathbf{x}, \sqrt{t}))};$$

$$(8.14) \quad \mathbf{H}_t(\mathbf{x}, \mathbf{y}) \leq \frac{C_2}{\mu(B(\mathbf{x}, \sqrt{t}))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, \sqrt{t}))}\right)^{-1-\delta};$$

$$(8.15) \quad |\mathbf{H}_t(\mathbf{x}, \mathbf{y}) - \mathbf{H}_t(\mathbf{x}, \mathbf{y}')| \leq \frac{C_4}{\mu(B(\mathbf{x}, \sqrt{t}))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, \sqrt{t}))}\right)^{-1-2\delta} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, \sqrt{t}))}\right)^{\frac{1}{N}}$$

for $\tilde{d}(\mathbf{y}, \mathbf{y}') \leq C_3 \max\{\mu(B(\mathbf{x}, \sqrt{t})), \tilde{d}(\mathbf{x}, \mathbf{y})\}$, (the kernel \mathbf{S}_t is denoted in [1] by \mathbf{P}_t). Moreover, the maximal function

$$\mathbf{S}_* f(x) = \sup_{t>0} \left| \int \mathbf{S}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mu(\mathbf{y}) \right|$$

is a bounded operator on $L^1(\mathbb{R}^n, d\mu)$.

Using subordination formula (2.1) we write

$$(8.16) \quad P_t(\mathbf{x}, \mathbf{y}) = U_t(\mathbf{x}, \mathbf{y}) + W_t(\mathbf{x}, \mathbf{y}),$$

where

$$U_t(\mathbf{x}, \mathbf{y}) = c_1 \int_0^\infty e^{-u} \mathbf{H}_{t^2/4u}(\mathbf{x}, \mathbf{y}) \frac{du}{\sqrt{u}}, \quad W_t(\mathbf{x}, \mathbf{y}) = c_1 \int_0^\infty e^{-u} \mathbf{S}_{t^2/4u}(\mathbf{x}, \mathbf{y}) \frac{du}{\sqrt{u}}.$$

Clearly, the maximal operator

$$W_* f(\mathbf{x}) = \sup_{t>0} \left| \int W_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mu(\mathbf{y}) \right|$$

is bounded on $L^1(\mathbb{R}^n, d\mu)$, that is,

$$(8.17) \quad \|W_* f\|_{L^1(\mathbb{R}^n, d\mu)} \leq C \|f\|_{L^1(\mathbb{R}^n, d\mu)}.$$

Our task is to prove the following lemma.

Lemma 8.18. *There are constants $C_1, C_2, C_4, \delta' > 0$ such that*

$$(8.19) \quad U_t(\mathbf{x}, \mathbf{x}) \geq \frac{C_1}{\mu(B(\mathbf{x}, t))};$$

$$(8.20) \quad U_t(\mathbf{x}, \mathbf{y}) \leq \frac{C_2}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta'};$$

$$(8.21) \quad |U_t(\mathbf{x}, \mathbf{y}) - U_t(\mathbf{x}, \mathbf{y}')| \leq \frac{C_4}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-2\delta'} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\delta'}$$

for $\tilde{d}(\mathbf{y}, \mathbf{y}') \leq C_3 \max\{\mu(B(\mathbf{x}, t)), \tilde{d}(\mathbf{x}, \mathbf{y})\}$.

Proof. Take $0 < \delta < \mathbf{N}^{-1}$. By (8.13) and the subordination formula we have

$$\begin{aligned} U_t(\mathbf{x}, \mathbf{x}) &\gtrsim \int_1^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}} \gtrsim \int_1^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}} \\ &\gtrsim \frac{1}{\mu(B(\mathbf{x}, t))} \int_1^\infty \frac{e^{-u}}{\sqrt{u}} du \gtrsim \frac{1}{\mu(B(\mathbf{x}, t))}, \end{aligned}$$

which proves (8.19).

The proof of (8.20) is similar to that of (8.9). Indeed, by (8.14) we have

$$\begin{aligned} U_t(\mathbf{x}, \mathbf{y}) &\leq \int_0^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}} \\ &\leq \int_0^{\frac{1}{4}} \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \left(\frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}} \\ &\quad + \int_{\frac{1}{4}}^\infty \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \frac{du}{\sqrt{u}}. \end{aligned}$$

Now using (8.5) we obtain (8.20).

Now we turn to the proof of (8.21). First we show that for every $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^n$ we see that

$$(8.22) \quad |U_t(\mathbf{x}, \mathbf{y}) - U_t(\mathbf{x}, \mathbf{y}')| \leq \frac{C_4}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}}.$$

Since $U_t(\mathbf{x}, \mathbf{y}) \leq C\mu(B(\mathbf{x}, t))^{-1}$ (see (8.20)), it suffices to prove (8.22) for $\tilde{d}(\mathbf{y}, \mathbf{y}') \leq \mu(B(\mathbf{x}, t))$. Let $u_0 \geq 1/4$ be such that $\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u_0}})) = \tilde{d}(\mathbf{y}, \mathbf{y}')$. Then, using (8.15) and (8.5), we have

$$\begin{aligned} &\int_0^{u_0} e^{-u} |\mathbf{H}_{\frac{t^2}{4u}}(\mathbf{x}, \mathbf{y}) - \mathbf{H}_{\frac{t^2}{4u}}(\mathbf{x}, \mathbf{y}')| \frac{du}{\sqrt{u}} \lesssim \int_0^{u_0} \frac{e^{-u}}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{\frac{1}{\mathbf{N}}} \frac{du}{\sqrt{u}} \\ &= \int_0^{u_0} \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}} \left(\frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))}\right)^{1+\frac{1}{\mathbf{N}}} \frac{du}{\sqrt{u}} \\ &\lesssim \frac{1}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}} \left(\int_0^{1/4} e^{-u} u^{n(1+\frac{1}{\mathbf{N}})/2} \frac{du}{\sqrt{u}} + \int_{1/4}^{u_0} e^{-u} u^{\mathbf{N}(1+\mathbf{N}^{-1})/2} \frac{du}{\sqrt{u}}\right) \\ &\lesssim \frac{1}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}}. \end{aligned}$$

Similarly, by (8.14), we get

$$\begin{aligned} \int_{u_0}^{\infty} e^{-u} |\mathbf{H}_{\frac{t^2}{4u}}(\mathbf{x}, \mathbf{y}) - \mathbf{H}_{\frac{t^2}{4u}}(\mathbf{x}, \mathbf{y}')| \frac{du}{\sqrt{u}} &\lesssim \int_{u_0}^{\infty} \frac{e^{-u}}{\mu(B(\mathbf{x}, t))} \left(\frac{\mu(B(\mathbf{x}, t))}{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u}}))} \right) \frac{du}{\sqrt{u}} \\ &\lesssim \frac{1}{\mu(B(\mathbf{x}, t))} \int_{u_0}^{\infty} e^{-u} u^{\mathbf{N}/2} \frac{du}{\sqrt{u}} \\ &\lesssim \frac{1}{\mu(B(\mathbf{x}, t))} u_0^{-\mathbf{N}/2}. \end{aligned}$$

Since

$$\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))} = \frac{\mu(B(\mathbf{x}, \frac{t}{2\sqrt{u_0}}))}{\mu(B(\mathbf{x}, t))} \gtrsim u_0^{-\mathbf{N}/2},$$

(see (8.5)), we obtain (8.22).

We are now in a position to continue the proof of (8.21).

If $\tilde{d}(\mathbf{x}, \mathbf{y}) \leq \mu(B(\mathbf{x}, t))$ then (8.21) follows from (8.22).

If $\tilde{d}(\mathbf{x}, \mathbf{y}) > \mu(B(\mathbf{x}, t))$ and $\tilde{d}(\mathbf{y}, \mathbf{y}') < \tilde{d}(\mathbf{x}, \mathbf{y})/(2A)$, then $\tilde{d}(\mathbf{x}, \mathbf{y}) \leq 2A\tilde{d}(\mathbf{x}, \mathbf{y}')$. Hence, from (8.20) we conclude that

$$(8.23) \quad |U_t(\mathbf{x}, \mathbf{y}) - U_t(\mathbf{x}, \mathbf{y}')| \leq \frac{C'_2}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta}.$$

Consequently, we deduce (8.21) (with perhaps small $\delta' > 0$) from (8.22) and (8.23).

It remains to consider the case when $\tilde{d}(\mathbf{x}, \mathbf{y}) > \mu(B(\mathbf{x}, t))$ and $\tilde{d}(\mathbf{y}, \mathbf{y}') \geq \tilde{d}(\mathbf{x}, \mathbf{y})/(2A)$. Recall that $\tilde{d}(\mathbf{y}, \mathbf{y}') \leq \mu(B(\mathbf{x}, t))$. Thus $\tilde{d}(\mathbf{x}, \mathbf{y}) \sim \mu(B(\mathbf{x}, t))$. So, finally, using (8.22) we have

$$|U_t(\mathbf{x}, \mathbf{y}) - U_t(\mathbf{x}, \mathbf{y}')| \leq \frac{C_4}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}} \leq \frac{C_4}{\mu(B(\mathbf{x}, t))} \left(\frac{\tilde{d}(\mathbf{y}, \mathbf{y}')}{\mu(B(\mathbf{x}, t))}\right)^{\frac{1}{\mathbf{N}}} \left(1 + \frac{\tilde{d}(\mathbf{x}, \mathbf{y})}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta}.$$

This completes the proof of Lemma 8.18. \square

Set $K_r(\mathbf{x}, \mathbf{y}) = U_t(\mathbf{x}, \mathbf{y})$, where $r = \mu(B(\mathbf{x}, t))$. Now part (a) of Theorem 2.4 follows from (8.16), boundedness of the maximal function W_* on $L^1(\mathbb{R}^n, \mu)$, and the Uchiyama theorem (see Theorem 8.11) combined with Lemma 8.18.

Now we turn to the proof of part (b) of Theorem 2.4. Recall that $P_t(\mathbf{x}, \mathbf{y}) > 0$. So, by (8.6), the operator P_* is bounded on $L^\infty(\mathbb{R}^n, d\mu)$. Thanks to (8.20) and (8.17), it is of weak-type (1,1). Finally, from the Marcinkiewicz interpolation theorem we conclude that P_* is bounded on $L^p(\mathbb{R}^n, d\mu)$ for $1 < p < \infty$. \square

Proof of Proposition 2.5. Fix $\varepsilon > 0$. There is $R > 0$ such that $|g(\mathbf{x})| < \varepsilon$ for $|\mathbf{x}| > R$. Write

$$g = g\chi_{B(0,R)} + g\chi_{B(0,R)^c} =: g_0 + g_1.$$

From (8.6) we get $|P_t g_1(\mathbf{x})| < \varepsilon$ for every $t > 0$ and $\mathbf{x} \in \mathbb{R}^n$. Now using (8.8) we obtain

$$|P_t g_0(\mathbf{x})| \leq \frac{C}{\mu(B(\mathbf{x}, t))} \|g_0\|_{L^1(\mathbb{R}^n, d\mu)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

On the other hand, if t remains in a bounded interval and $|\mathbf{x}| > 2nR$, applying (8.9) we have

$$|P_t g_0(\mathbf{x})| \leq \frac{C}{\mu(B(\mathbf{x}, t))} \left(1 + \frac{\mu(B(\mathbf{x}, |\mathbf{x}|))}{\mu(B(\mathbf{x}, t))}\right)^{-1-\delta} \|g_0\|_{L^1(\mathbb{R}^n, d\mu)} \rightarrow 0 \quad \text{as } |\mathbf{x}| \rightarrow \infty.$$

The proof of the first part of Proposition 2.5 is complete.

In order to prove the second part of the proposition we fix $\varepsilon > 0$. We claim that

$$\lim_{|\mathbf{x}| \rightarrow \infty} P_\varepsilon f(\mathbf{x}) = 0.$$

To prove the claim let $\varepsilon' > 0$. Take $R > 0$ large enough such that $\int_{|y|>R} |f(y)| d\mu(y) \leq \varepsilon' \mu(B(0, \varepsilon))$. Write $f = f\chi_{B(0,R)} + f\chi_{B(0,R)^c} =: f_0 + f_1$. Then, by (8.6) and (8.8) we have $|P_\varepsilon f_1| \leq \varepsilon'$. On the other hand from the first part of the proposition we conclude that $\lim_{|x| \rightarrow \infty} P_\varepsilon f_0(x) = 0$, which gives the claim. Now (2.6) follows from the first part of Proposition 2.5, since $P_{t+\varepsilon} f = P_t(P_\varepsilon f)$. \square

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